STOCHASTIC HOMOGENIZATION OF STEEL FIBER REINFORCED CONCRETE

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Abstract

In recent papers of the author, deterministic models depending on the distribution of fibers in composite structures have been studied. For example, optimization, homogenization, localization, etc., have been solved. The extended Hashin-Shtrikman variational principles, eigenparameters in which were involved in the formulations, served as the starting point, and the comparative medium was introduced. An integral formulations were then derived form the principles admissible and efficient. The formulatons of the above-said problems require the restriction of geometry of the fibers to certain "locally reasonable" structures, e.g. to periodic unit cells, which were cut out of the representative volume element. Since the condition of regular distribution of fibers is violated in applications, and statistical (randomly) distribution is more probable, other extension of the H-S principles is needed. In this paper, the problem is extended to the case of statistically distributed fibers. Hashin-Shtrikman variational principles are formulated in terms of statistical characteristics in the domain and the eigenparameters are also involved, burdened by the statistical values. Following the H-S principles, integral formulation is stated (again, thanks to the use of the comparative medium such a formulation is admissible in a similar way as that in the deterministic formulation) in a representative volume element, which has no longer regular geometry of the fibers. The boundary element method has then a special form, which is advantageous particularly for multi-phase media. The above-mentioned formulation of Hashin-Shtrikman variational principles with randomly distributed fields of fibers can be extended to non-linear problems (plasticity, debonding) by introducing transformation fields (eigenstresses or eigenstrains, which are involved in the formulations). From a wide range of applications of the theory, a homogenization of material properties of the fiber reinforced concrete is studied. Note that some of approaches from classical composites will be applied. From a large list of publications a couple of papers is introduced in References. Such papers provide basic information on overall material properties assuming knowledge of heterogeneous material and geometry on micro level.

Keywords: Fiber reinforced concrete, stochastic analysis, overall properties, Dramix type of fibers, boundary element method
1. Introduction

Eigenstresses and eigenstrains play a very important role in many branches of applied mechanics, e.g., in composites, geotechnics, concrete structures, etc. In previous papers, e.g., [7], the authors have formulated an effective approach to the analysis and optimization of nonhomogeneous bodies with prescribed boundary displacements, or tractions and have used the transformation field analysis for relating the components of stress or strain tensors and the components of eigenstresses or eigenstrains. The transformation field analysis established by Dvorak in a special form. The generalization to structures has larger possibility in applications and we used and use the same denotation for this generalized form. The confusion does not threat. The new form was introduced by Dvorak and Procházka in [1], and was there applied to localization of stresses and strains in two-phase composites. The eigenstresses stood for relaxation stresses and eigenstrains represented plastic strains. This idea was particularly used in [2], where a large scale of combinations of material properties together with prestresses of composite structures was considered. In [1], thick-walled cylindrical structures were studied while in [2] submerged cylindrical laminates with different properties in combination with prestress were discussed.

Using the eigenparameters in the sense of [1], [7] and generalize it to the macrostructure (homogenization) of composites, one can obtain procedures that involve very wide range of nonlinear problems (plasticity, viscoplasticity, damage, etc.). This is why we are interested in such variational formulations that are valid for various composites. To this end the Hashin-Shtrikman variational principles, [4], are the most appropriate means for this intention. The Hashin-Shtrikman variational principles have been applied to explicit estimation of material bounds in [5], for example. The extended H-S principles are precised in [8]. An integral formulation can be formulated, [7], and the boundary element method is then applicable to obtain not only bounds of material properties, but also for very precise homogenization of them. In comparison with the finite element method the boundary element method appears to be far more efficient in this case.

It is worth noting that the eigenparameters are generalization of change of temperature (eigenstrain). Such a problem was discussed in the well-known paper by Levin, [6].

Our approach is based on the idea of extended Hashin-Shtrikman variational principles, involving both the eigenparameters and randomly distributed phases. By means of internal parameters, eigenstrains or eigenstresses, involved in the H-S principles, it is possible to compute overall material properties of the trial material, increase the bearing capacity of structures, or to minimize the stress excesses.

In the current time the problems of reliability of materials are relatively marginal, as in the practice the standards provide results based on reliability theory in a comprehensive form. A material with randomly distributed phases is discussed by Drugan and Willis, [3], in connection with H-S principles, defined on unbounded domain. Based on the finite element method, the stochastic homogenization is discussed in [9].

2. Extended Hashin Shtrikman variational principles

In this section we concentrate only on the first H-S principle. Suppose that no body forces are present. Let us consider a bounded domain \( \Omega \) with bounded Lipschitz's boundary \( \Gamma \) and with subdomains \( \Omega_i \subseteq \Omega, i = 1, \ldots, n \), describing local inclusions. Two steps in constructing the formulation are distinguished: First, let \( \epsilon_{ij}^0 \) and \( \sigma_{ijkl}^0 \) be the known strain and stress fields, respectively. They are related by Hooke's law as:

\[
\sigma_{ijkl}^0 = E_{ijkl} \epsilon_{ij}^0, \quad \phi_{ij}^0 = M_{ijkl} \sigma_{kl}^0
\]  

(1)
where $L_{ijkl}^0$ and $M_{ijkl}^0$ are material stiffness and compliance tensor of elastic constants given for comparative medium. Subscripts in (1) run $1, \ldots, 3$ in 3D. Boundary conditions for real problem are fulfilled.

In the second step, the real heterogeneous medium is considered, which is geometrically identical as the comparative medium and the boundary conditions, whether statical or geometrical, are the same as in the first case. Components of displacement vector $u_i$, strains tensor $\varepsilon_{ij}$ and stress tensor $\sigma_{ij}$ are unknown and generalized Hooke's law including the eigenstresses $\lambda_{ij}$ and eigenstrains $\mu_{ij}$ can be written as:

$$\sigma_{ij} = L_{ijkl}^0 \varepsilon_{kl} + \lambda_{ij}, \quad \varepsilon_{ij} = M_{ijkl}^0 \sigma_{kl} + \mu_{ij}$$  \hfill (2)

Comparing both equivalent expressions for Hooke's law one obtains: $\lambda_{ij} = -L_{ijkl}^0 \mu_{ij}$.

Similarly to the classical Hashin-Shtrikman principles define the symmetric stress polarization tensor with components $\tau_{ij}$ by:

$$\tau_{ij} = [L_{ijkl}^0] \varepsilon_{kl} + \lambda_{ij}, \quad [L_{ijkl}] = L_{ijkl}^0 - L_{ijkl}^0$$  \hfill (3)

Also define

$$\begin{align*}
    u'_i &= u_i - u_i^0, \\
    \varepsilon'_{ij} &= \varepsilon_{ij} - \varepsilon_{ij}^0, \\
    \sigma'_{ij} &= \sigma_{ij} - \sigma_{ij}^0
\end{align*}$$  \hfill (4)

Subtracting (1) from (2) and using definition (3) gives:

$$\sigma'_{ij} = L_{ijkl}^0 \varepsilon'_{kl} + \tau_{ij}$$  \hfill (5)

Since both $\sigma_{ij}$ and $\sigma_{ij}^0$ are statically admissible, taking into consideration (5) the following equation has to be satisfied in the sense of distributions:

$$\frac{\partial \sigma'_{ij}}{\partial x_j} = \frac{\partial (L_{ijkl}^0 \varepsilon'_{kl} + \tau_{ij})}{\partial x_j}, \quad u'_i = 0$$  \hfill (6)

Assuming (5), (6) and (3), the primary extended H-S variational principle, may be obtained by seeking the stationary point of the extended functional $U$ defined as:

$$U = U^0 - \frac{1}{2} \int_{\Omega} \left( \left( [L_{ijkl}^0]^{-1} (\tau_{ij} - \lambda_{ij})(\tau_{kl} - \lambda_{kl}) - 2 \tau_{ij} \varepsilon_{ij}^0 - \varepsilon'_{ij} \tau_{ij} - M_{ijkl}^0 \lambda_{ij} \lambda_{kl} \right) \right) d\Omega$$  \hfill (7)

with respect to the fields $\tau_{ij}$ and $\varepsilon'_{ij}$. In (7) we have denoted:

$$U^0 = \frac{1}{2} \int_{\Omega} \sigma_{ij}^0 \varepsilon_{ij}^0 \, d\Omega = \frac{1}{2} \int_{\Omega} L_{ijkl}^0 \varepsilon_{ij}^0 \varepsilon_{ij}^0 \, d\Omega$$

The second H-S principle can be derived in a similar manner.
The bounds on elastic material properties are derived from certain additional presumptions. Since we do not need them, we drop this out. For more details, see [8]. Some corrections of the original assumptions on the properties of the second Gateau derivatives are there specified. Note that the unbounded three-dimensional composite structure, considered in [3] does not admit 2D formulation because of singularity of kernels used. In 3D, the radiation condition, assuming the unicity of the solution, be valid, and the formulation is correct. When using integral formulation on bounded domain this danger does not threat and 2D formulation can easily be used. In what follows the eigenparameters will not be considered explicitly, for the sake of simplicity, but will be involved in the polarization tensor. The primary H-S principle will be discussed; the dual can be derived in a similar way.

An integral equation equivalent to (7) is (see [7]):

\[
 u'_i(\xi) = \int_{\Gamma} [\mu^*_{ik}(x, \xi)p^*_k(x) - \mu^*_{ik}(x, \xi)u^*_k(x)] d\Gamma(x) - \int_{\Omega} \varepsilon_{ikl}^*(x, \xi)\tau_{kl}(x) d\Omega(x), \quad u'_i(\xi) \equiv 0 \quad \text{on} \quad \Gamma \quad (8)
\]

where the starred quantities are known kernels.

Differentiating the last equation with respect to \( \xi \) provides the expression:

\[
 \varepsilon'_{ij}(\xi) = \int_{\Gamma} [\mu^*_{ik}(x, \xi)p^*_k(x) - \mu^*_{ik}(x, \xi)u^*_k(x)] d\Gamma(x) - \int_{\Omega} \psi_{ikl}^*(x, \xi)\tau_{kl}(x) d\Omega(x), \quad u'_i(\xi) \equiv 0 \quad \text{on} \quad \Gamma \quad (9)
\]

and the volume integral in (8) and (9) are taken in Hadamard's sense.

The deterministic solution of (8), (9) is relatively easy providing that the geometry of the composite aggregate and the material properties are known. It is worth pointing out that the H-S principles enabled us to use the above integral formulation (and, consequently, to apply the boundary element method). The H-S principles are also necessary in our following study of randomly distributed phases.

3. Randomly distributed phases

In what follows, we generalize the formulation of the deterministic problem of localization and homogenization of composite aggregate problem to randomly distributed phases. A typical micropicture is shown in Fig. 1.

Let from now on the fibers are randomly distributed in the composite medium. Denote by \( \alpha \) individual phenomena in a sample space S. The probability density of \( \alpha \) in S be defined by \( d(\alpha) \). Characteristic function \( \kappa_s(\xi, \alpha) \) is equal to 1 when \( \xi \) lies in the phase \( \Omega \), and to 0 otherwise. Then the probability \( P_r(\xi) \) of finding the phase \( \Omega_r \) at \( \xi \) is the ensemble average of \( \kappa_r(\xi, \alpha) \), i.e.,

\[
 P_r(\xi) = \langle \kappa_r(\xi, \alpha) \rangle = \int_S \kappa_r(\xi, \alpha) d(\alpha) \quad d\alpha \quad (10)
\]

Two-point probability of finding the phase \( \Omega_r \) at \( \xi \) and \( \Omega_s \) at \( x \) is defined by:

\[
 P_{rs}(\xi, x) = \langle \kappa_r(\xi, \alpha), \kappa_s(\xi, \alpha) \rangle = \int_S \kappa_r(\xi, \alpha)\kappa_s(\xi, \alpha) d(\alpha) \quad d\alpha \quad (11)
\]

Further assume that the material is statistically uniform, i.e., the probability and two-point probability are insensitive to translation. This assumption is reasonable in many cases of materials, and we approximately can introduce it here, too, for simplicity. Then \( P_r(\xi) = c_r \), the volume fraction ratio of the phase \( \Omega_r \), and \( P_{rs}(\xi, x) = P_{rs}(||\xi - x||) \).

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From now on we rather use the tensor notation with boldface variables, for the sake of brevity. Since the variables in our problem now undergo statistical laws, we write: the stiffness matrix $L(\xi,\alpha)$, the stress $\sigma(\xi,\alpha)$, the strain $\varepsilon(\xi,\alpha)$, and displacements $u(\xi,\alpha)$. On the other hand, this is not valid for the internal fields in the comparative medium and it still remains dependency $\sigma^0(\xi), \varepsilon^0(\xi), u^0(\xi)$.

Fig. 1: Microphoto of a typical FRC aggregate, fiber volume ratio equals 0.6

The stiffness matrix and its ensemble average are calculated as:

$$L(\xi;\alpha) = \sum_{r=1}^{n} L' \kappa_r(\xi;\alpha), \quad <L(\xi;\alpha)> = \sum_{r=1}^{n} L' P_r(\xi) = \sum_{r=1}^{n} L' c_r$$ (12)

Assume a general field $\tau(\xi, x;\alpha)$ be restricted to one- and two-point correlations. Moreover, three phase medium will be considered: $\Omega_c \subset \Omega$ is the subdomain of the structural element (representative volume element – RVE), describing exceptionally the concrete skeleton without voids and fiber, $\Omega_f \subset \Omega$ is the subdomain describing only fibers, their volume fiber ratio is very low, e.g. 0.5 – 1 percent, the distribution of unknowns is approximated by their averages, and eventually $\Omega_v \subset \Omega$ is the subdomain covering voids, where no stress exists (the situation concerning displacements is not solved here, we do not need to care about it), only pore pressure can be defined. Since the volume fraction ratio is even less than that of fibers (0.1 percent or so), the unknown quantities are again constant.

Let us come back to the formula (3) defining the polarization tensors. Let us select the comparative medium to be concrete matrix. Then we get:
\[ \tau^0 = 0, \quad \tau^f = (L^f - L^0) \varepsilon \approx \text{const.}, \quad \tau^v = \lambda - L^0 \varepsilon' \] (13)

where in the last formula (13) has been used and also \( \sigma^0 = 0 \) inside the void. Moreover the pore pressure is spread out uniformly, and consequently, also the last term is a constant. Sum the last results up we arrive to the result as \( (r = 1 \text{ for fibers and } r = 2 \text{ for voids}) \):

\[ \tau(\xi; \alpha) = \sum_{r=1}^{2} \tau^r E_r(\xi; \alpha) \] (14)

Positioning \( \xi \) to the boundary \( \Gamma \), eq. (8) in stochastic version can be recorded as:

\[ \int_{\Gamma} u_{ik}^*(x, \xi)p_{ik}^*(x, \alpha) \, d\Gamma(x) = \sum_{r=1}^{2} \tau_{ik}^* \int_{\Omega_r} \varepsilon_{ikl}^*(x, \xi) \kappa_r(x, \alpha) \, d\Omega(x), \] (15)

Setting boundary element and internal cell approximation as:

\[ p'(x, \alpha) = N^T(x)P'(\alpha) \] (16)

and similarly for internal cells, substituting the approximation in (15) yields:

\[ UP'(\alpha) = \sum_{r=1}^{2} \tau^r \int_{\Omega_r} \varepsilon_{ik}^*(x, \xi) \kappa_r(x, \alpha) \, d\Omega(x) = \sum_{r=1}^{2} \tau^r E_r^*(\alpha) \] (17)

Note that \( V \) is a square matrix (DOF x DOF), \( P' \) is a vector of unknowns relative tractions (DOF), and so is \( E_r^* \) for each stochastic state and in each subdomain \( \Omega_r \). DOF is degrees of freedom on the boundary. \( E_r^*(\alpha) \) is a formal denotation, the kernel \( \varepsilon_{ikl}^* \) cannot depend on a statistical case. It is well known from the theory of the boundary element method that \( U \) must be regular and, consequently, invertible, i.e. admits its inverse \( U^{-1} \). Using the expression for approximation of the relative tractions (16) and applying it to (17) gives:

\[ p'(x, \alpha) = N^T(x)U^{-1}\sum_{r=1}^{2} \tau^r E_r^*(\alpha) \] (18)

Our aim in the next step will be to eliminate the unknown relative boundary tractions from the formula (9). Note that the relative deformations \( \varepsilon^* \) depends also on the stochastic state \( \alpha \). Substituting (18) to (9) provides:

\[ \varepsilon_{ij}^*(\xi, \alpha) = \int_{\Gamma} h_{ijk}^*(x, \xi)p_{ik}^*(x, \alpha) \, d\Gamma(x) - \sum_{r=1}^{2} \tau^r \int_{\Omega_r} \psi_{ik}^*(x, \xi) \kappa_r(x, \alpha) \, d\Omega(x) = \]

\[ = \int_{\Gamma} H^*N^T(x) \, d\Gamma(x)U^{-1}\sum_{r=1}^{2} \tau^r E_r^*(\alpha) - \sum_{r=1}^{2} \tau^r \Psi_r^*(\alpha) \] (19)

where

\[ \sum_{r=1}^{2} \tau^r \Psi_r^*(\alpha) = \sum_{r=1}^{2} \tau^r \int_{\Omega_r} \psi^*(x, \xi) \kappa_r(x, \alpha) \, d\Omega(x) \]
and similarly for $\Psi_i^r(\alpha)$.

In order to separate quantities depending on stochastic processes and the deterministic ones, multiplying \((19)\) by $T(\xi; \alpha)$ over $\Omega$ one obtains:

$$\int_{\Omega} \tau(x; \alpha) \varepsilon'(\xi, \alpha) \, d\Omega(\xi) = \sum_{r=1}^{2} \sum_{s=1}^{2} \int \int G(x, \xi) \tau' \kappa_r(x, \alpha) \kappa_s(\xi, \alpha) \, d\Omega(x) \, d\Omega(\xi)$$

(20)

where

$$G(x, \xi) = \Psi^r(x, \xi) - \int_{\Gamma} H^r(x) \, d\Gamma(x) U^{-1} E^r(x)$$

Substituting \((20)\) to \((7)\) and carry out the assembly averaging we obtain the H-S principle for $\tau^s$ as:

$$<U> = <U_0> - \frac{1}{2} \sum_{r=1}^{2} \sum_{s=1}^{2} \int \int (\tau' - \lambda)^T [L_r - L^{0 \, T}] (\tau' - \lambda)$$

$$- 2(\tau')^T \varepsilon'(x) - \lambda \tau M^{\tau} \lambda \, d\Omega_r - \sum_{r=1}^{2} \sum_{s=1}^{2} \int \int G(x, \xi) \varepsilon' \tau' P_{rs}(x, \alpha) \, d\Omega(x) \, d\Omega(\xi)$$

(21)

In this case the advantage of the boundary elements consists in very coarse discretization of the boundary, choice of two-point probability as exp distance function and performance of the integrals over domains with uniformly distributed quantities as their representative at the centers of domains.

Minimization of the latter functional with respect to $\tau$ leads to two unknown polarization tensors for fibers and voids.

Note that the two-point probability uses exponential distant function. 2D problem is solved. The material properties of phases are introduced in Table 1. The results of stochastic homogenization are shown in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>$E$(GPa)</th>
<th>$v$</th>
<th>$c$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber</td>
<td>210</td>
<td>0.16</td>
<td>0.6</td>
</tr>
<tr>
<td>Concrete</td>
<td>16.6</td>
<td>0.3</td>
<td>99.2</td>
</tr>
</tbody>
</table>

Table 1 Material properties of the phases

<table>
<thead>
<tr>
<th>$L_{11}$</th>
<th>$L_{22}$</th>
<th>$L_{12}$</th>
<th>$L_{33}$</th>
<th>$c_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.5</td>
<td>18.5</td>
<td>3.92</td>
<td>7.58</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Table 2 Resulting stiffness tensor
4. Conclusions

The extended Hashin-Shtrikman principles involving eigenstresses (representing the pore pressure in voids occurring in the concrete) together with the application equation application may describe overall properties of composite structure of the type of the fiber reinforce concrete. In this paper linear behavior of both fibers and concrete matrix is considered together with pore pressure in void remaining in the aggregate.

From the results obtained in this paper some interesting conclusions follow. It is seen that the type of boundary conditions of RYE does not play any important role. This was observed also in [3] and [9] where comparison with the method by Mori-Tanaka was carried out and the error with respect to M-T method was negligible. The M-T method starts with an assumption that prevailing influence of the matrix is decisive, no matter the boundary conditions are.

In comparison with the FEM, [3], [9], the boundary elements are much more promising, as they concentrate the DOF only on the boundary and do not care about the interior of the domain of RVE. In the final stage of statistically homogeneous medium is assumed (the BEM naturally admissible). Note that the boundary element method reduces the space of the problem by one.

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4. References


